

Resonances of the cusp family

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Abstract

We study a family of chaotic maps with limit cases the tent map and the cusp map (the cusp family). We discuss the spectral properties of the corresponding Frobenius–Perron operator in different function spaces including spaces of analytic functions. A numerical study of the eigenvalues and eigenfunctions is performed.

1 Introduction

Resonances of dynamical systems play important role in the study of the decay of correlations and are manifestations of the statistical properties of chaotic systems. Hence it is not surprising that they are studied very intensively. We refer readers to recent reviews on this vast subject [1, 2]. Resonances appear also in the generalized spectra [3, 4, 5] of the evolution operators [6, 7] of chaotic maps.

The theory of resonances has been recently developed in terms of locally convex topological vector spaces [3, 4, 5]. This reflects the fact that dynamical systems are defined in terms of the space of observables and the evolution law. For different classes of observables the same evolution law may have different resonances i.e. different rates of approach to equilibrium. However once the class of observables is chosen the resonance structure is unique [5, 8]. Therefore we have proposed [5, 8, 9] that physical equivalence should reflect identical physical properties i.e. rates of decay of correlations.

For many classes of maps, e.g. expanding maps, there exist some exact results about existence of resonances and their estimations [2]. However, for more complicated maps each case needs a separate consideration and results are sparse. Their study has attracted a lot of interest.

For example, the so-called cusp map [10]

$$F : [-1, 1] \rightarrow [-1, 1], \quad \text{where} \quad F(x) = 1 - 2\sqrt{|x|}$$

is an approximation of the Poincaré section of the Lorenz attractor [11, 12]. The absolutely continuous invariant probability measure of the cusp map has density

$$\rho(x) = \frac{1-x}{2}.$$

The cusp map is a limit case of the cusp family [13, 14, 15]:

$$\begin{aligned} F_\varepsilon &: [-1, 1] \rightarrow [-1, 1], \quad \varepsilon \in [0, 1/2], \quad \text{where} \\ F_\varepsilon(x) &= \frac{1 - \sqrt{1 - 4\varepsilon(1 - \varepsilon - 2|x|)}}{2\varepsilon} \quad \text{for } \varepsilon \in (0, 1/2], \\ F_0(x) &= \lim_{\varepsilon \downarrow 0} F_\varepsilon(x) = 1 - 2|x|. \end{aligned} \tag{1}$$

The map with $\varepsilon = 0$ is the well-known tent map [16] while the map with $\varepsilon = 1/2$ is the cusp map [10].

Each map in the cusp family is an exact system. For $\varepsilon \neq 1/2$ it follows directly from theorem 4 in § 8, chap. 10 of [7], which gives sufficient conditions of exactness for piecewise monotonic maps. The exactness of the cusp map has been hinted by Hemmer [10] referring to the work of Lasota and Yorke [17]. The proposed hint seems to be irrelevant as the cusp map has a parabolic fixed point.

For the cusp map one should consider the so-called induced map [18, 19, 20] on the segment $[\sqrt{8} - 3, 3 - \sqrt{8}]$. This map satisfies the conditions of the above mentioned theorem and therefore is exact. Since the exactness for a map and its induced map are equivalent [18, 19, 20], we obtain the exactness of the cusp map.

The unique absolutely continuous Borel invariant probability measure μ_ε for the cusp family F_ε has density [13]

$$\rho_\varepsilon(x) = \frac{1}{2} - \varepsilon x. \tag{2}$$

The statistical analysis of dynamical systems is based on the Koopman and the Frobenius–Perron operators. The Koopman operator of a measurable map $S : Y \rightarrow Y$, where (Y, \mathcal{F}) is a measurable space, acts on functions $f : Y \rightarrow \mathbf{C}$:

$$Vf(x) = f(Sx).$$

The Frobenius–Perron operator (F.P.O.) U is defined with respect to a probability reference measure ν on (Y, \mathcal{F}) . For $1 \leq p \leq \infty$ the F.P.O. $U : L_p(Y, \mathcal{F}, \nu) \rightarrow L_p(Y, \mathcal{F}, \nu)$ is the dual of the operator $V : L_q(Y, \mathcal{F}, \nu) \rightarrow L_q(Y, \mathcal{F}, \nu)$, where $\frac{1}{p} + \frac{1}{q} = 1$:

$$(U\rho|f) = (\rho|Vf), \quad (\rho|f) = \int \nu(dy) \overline{\rho(y)} f(y).$$

In case of an exact endomorphism S on a segment $Y = [a, b]$ one usually use either the normalized Lebesgue measure or the invariant absolutely continuous probability measure as the reference measure. In both cases 1 is an eigenvalue of the F.P.O. However, in the first case the corresponding eigenfunction is the density of the invariant measure; in the second case the corresponding eigenfunction is constant **1**. In our paper we use the invariant measure as the reference one.

The Frobenius–Perron operator U_ε of F_ε with respect to the invariant measure μ_ε is

$$U_\varepsilon \rho(x) = \left(\frac{1}{2} - \varepsilon a_\varepsilon(x) \right) \rho(a_\varepsilon(x)) + \left(\frac{1}{2} + \varepsilon a_\varepsilon(x) \right) \rho(-a_\varepsilon(x)), \tag{3}$$

where

$$a_\varepsilon(x) = \frac{1-x}{2} - \frac{\varepsilon}{2}(1-x^2).$$

The objective of this paper is to study the resonances of the cusp family (1). In Section 2 we present some definitions and results for the spectral theory of operators necessary for the study of the F.P.O. of the cusp family. In Section 3 we present results about the spectral properties of the F.P.O. generated by this family in different function spaces. In Section 4 we analyze the spectral properties in spaces of analytic functions. In order to analyze the eigenvalues and eigenfunctions of the cusp family, we perform in Section 5 a numerical study. We show that the cusp family does not have spectrum in the form r^n , where $n \in \mathbf{N}$, $r \in \mathbf{R}$, in the space of analytic functions, at least in the vicinity of the tent map. We analyze the behavior of the eigenvalues in the vicinity of the cusp map. The behavior of the eigenfunctions is also discussed.

2 Normal points of linear operators

Let A be a linear continuous operator in a locally convex topological linear space E . The point $z \in \mathbf{C}$ is said to be *regular* if $A - zI$ has continuous inverse (here and below I is the identity operator). The set of all nonregular points is the spectrum of A , denoted as $\sigma(A)$. The point $z \in \mathbf{C}$ is said to be a *normal point* [21] if E admits a decomposition into a topological direct sum [22] of two closed linear subspaces

$$E = E_0 \oplus E_1 \tag{4}$$

such that E_0 is finite dimensional, $A(E_j) \subseteq E_j$ for $j \in \{0, 1\}$, $(A - zI)|_{E_1} : E_1 \rightarrow E_1$ has continuous inverse and there exists $n \in \mathbf{N}$ such that $(A - zI)^n(E_0) = \{0\}$.

Evidently the point z is regular if and only if it is normal and $E_0 = \{0\}$. A normal point for which $E_0 \neq \{0\}$ is called a *normal eigenvalue*.

It is well-known [21, 23] that for any normal point z the decomposition (4) is unique. Moreover, the monotonic sequences of spaces $\ker(A - zI)^n$ and $(A - zI)^n(E)$ stabilize and

$$E_0 = \bigcup_{n=1}^{\infty} \ker(A - zI)^n, \quad E_1 = \bigcap_{n=1}^{\infty} (A - zI)^n(E). \tag{5}$$

For a normal point z we denote

$$\mathcal{E}(z, A) = E_0. \tag{6}$$

Note that if z is regular then $\mathcal{E}(z, A) = \{0\}$. According to (5) the finite dimensional space $\mathcal{E}(z, A)$ is spanned by the eigenvectors and the principal vectors of A associated to the eigenvalue z . In the case $\mathcal{E}(z, A) \neq \{0\}$ the dimension of $\mathcal{E}(z, A)$ is the *multiplicity* of the normal eigenvalue z .

If the spectrum of A is either finite or is a sequence converging to 0 and any non-zero element of $\sigma(A)$ is a normal eigenvalue of A (this happens e.g. for any compact operator

on a Banach space [23]), we can relabel the spectrum $\sigma(A)$ as a sequence

$$z_n(A), \quad n = 0, 1, 2, \dots$$

so that the following conditions are fulfilled:

- 1) $|z_{n+1}(A)| \leq |z_n(A)|$ for all $n \in \mathbf{Z}_+$,
 - 2) if $z_n(A) \neq 0$ then $z_n(A) \in \sigma(A)$,
 - 3) if $z \in \sigma(A) \setminus \{0\}$ then $|\{n \in \mathbf{Z}_+ : z_n(A) = z\}| = \dim \mathcal{E}(z, A)$
 - 4) if $|z_n(A)| = |z_{n+1}(A)|$ then $\arg z_n(A) < \arg z_{n+1}(A)$,
- (7)

where $\arg z \in (-\pi, \pi]$ is the argument of the complex number z .

3 Spectral properties of the Frobenius–Perron operator in L_p and C^k

Let us introduce the following notation:

$$\overline{D}(a, q) = \{z \in \mathbf{C} : |z - a| \leq q\}, \quad D(a, q) = \{z \in \mathbf{C} : |z - a| < q\}, \quad (8)$$

For any $p \in [1, +\infty]$ we denote the Hardy space in the disk $D(a, q)$ by $\mathcal{H}^p(a, q)$, i.e. $\mathcal{H}^p(a, q)$ is the space of holomorphic functions $f : D(a, q) \rightarrow \mathbf{C}$, which belong to $L^p(D(a, q))$ with respect to the Lebesgue measure. We endow this space with the L_p -norm.

The operator $U_\varepsilon^X : X \rightarrow X$ is the restriction of U_ε to a locally convex function space X such that $U_\varepsilon(X) \subseteq X$. The spectrum of the operator U_ε^X is denoted by $\sigma(U_\varepsilon^X)$.

Proposition 1. *Let $\varepsilon \in [0, 1/2]$, X be either the Banach space $C[-1, 1]$ or $L_p([-1, 1], \mu_\varepsilon)$. Then the spectrum $\sigma(U_\varepsilon^X)$ coincides with the closed unit disk $\overline{D}(1)$. Moreover, any z from the open unit disk $D(1)$ is an eigenvalue of U_ε^X of infinite multiplicity. The point $z = 1$ is an eigenvalue of multiplicity 1.*

Proof. Since $\|U_\varepsilon^X\| = 1$ we have $\sigma(U_\varepsilon^X) \subseteq \overline{D}(1)$. Let $z \in D(1)$. Consider the Koopman operator of the cusp family $V_\varepsilon : X \rightarrow X$

$$V_\varepsilon f(x) = f(F_\varepsilon(x)).$$

One can directly verify that the functions ψ ,

$$\psi(x) = \sum_{k=0}^{\infty} z^k V_\varepsilon^k h, \quad (9)$$

where $h(x) = g(x)(1 + 2\varepsilon x)$ and $g(x)$ is an odd function, are eigenfunctions of U_ε : $U_\varepsilon \psi = z\psi$. As g is an arbitrary odd function, this proves that all points of $D(1)$ are eigenvalues of U_ε^X of infinite multiplicity. \square

Remark 1. Formula (9) provides all the eigenfunctions of U_ε^X with eigenvalue z .

Remark 2. Proposition 1 and its proof remain valid for the Frobenius–Perron operator U of any continuous exact endomorphism (instead of h one should take any element of $\ker U$).

Proposition 2. Let $\varepsilon \in [0, 1/2]$, $n = 1, 2, \dots$, X be the Banach space $C^n[-1, 1]$. Then the spectrum $\sigma(U_\varepsilon^X)$ contains the closed disk $\overline{D}(0, (1/2 + \varepsilon)^{n+1})$, and any point of the open disk $D(0, (1/2 + \varepsilon)^{n+1})$ is a (non-normal) eigenvalue of U_ε^X of infinite multiplicity. The set $S = \sigma(U_\varepsilon^X) \setminus \overline{D}(0, (1/2 + \varepsilon)^n)$ is finite and any $z \in S$ is a normal eigenvalue of U_ε^X .

Remark 3. Under the conditions of Proposition 2 for any $z \in S$ and any $f \in \mathcal{E}(z, U_\varepsilon^X)$ the function f admits the analytic continuation to the disk $D(0, 1/\varepsilon - 1)$ if $\varepsilon \in (0, 1/2)$ and to the whole complex plane if $\varepsilon = 0$. This can be proved by estimating the growth of the sequence $s_n = \sup_{t \in [-1, 1]} |f^{(n)}(t)|$.

Corollary 1. Let $\varepsilon \in [0, 1/2)$, $X = C^\infty[-1, 1]$ with the natural topology [22]. Then the spectrum $\sigma(U_\varepsilon^X)$ is either finite or countable, $0 \in \sigma(U_\varepsilon^X)$ and any point $z \in \sigma(U_\varepsilon^X) \setminus \{0\}$ is a normal (and therefore isolated) eigenvalue of U_ε^X . Moreover for any $z \in \sigma(U_\varepsilon^X) \setminus \{0\}$ and any $f \in \mathcal{E}(z, U_\varepsilon^X)$ the function f admits the analytic continuation to the disk $D(0, 1/\varepsilon - 1)$ if $\varepsilon \in (0, 1/2)$ and to the whole complex plane if $\varepsilon = 0$.

Corollary 2. Let $n = 1, 2, \dots, \infty$ and X be the space $C^n[-1, 1]$. Then the spectrum $\sigma(U_{1/2}^X)$ is the closed unit disk $\overline{D}(0, 1)$, and any point of the open unit disk $D(0, 1)$ is an eigenvalue of $U_{1/2}^X$ of infinite multiplicity.

Proof of Proposition 2.

Let us define the sequence t_n by the formula

$$t_0 = 1, \quad t_{n+1} = -a_\varepsilon(t_n), \quad n = 1, 2, \dots \quad (10)$$

It is easy to see that $t_1 = 0$, the sequence t_n is strictly decreasing and

$$\begin{aligned} t_n &= -1 + \frac{4}{n} + O\left(\frac{1}{n^2}\right) \quad \text{for } \varepsilon = 1/2, \\ t_n &= -1 + c(\varepsilon)\left(\frac{1}{2} + \varepsilon\right)^n + O\left(\left(\frac{1}{2} + \varepsilon\right)^{2n}\right), \quad \text{for } \varepsilon \in [0, 1/2). \end{aligned} \quad (11)$$

Let $z \in \mathbf{C}$. Pick an arbitrary function $\phi : (0, 1] \rightarrow \mathbf{C}$. Define recurrently the function $f_\phi : (-1, 1] \rightarrow \mathbf{C}$ as follows

$$\begin{aligned} f_\phi(x) &= \phi(x) \quad \text{for } x \in (0, 1] = (t_1, t_0], \\ f_\phi(x) &= \frac{2zf(a_\varepsilon^{-1}(-x))}{1-2\varepsilon x} - \frac{1+2\varepsilon x}{1-2\varepsilon x} f(-x) \quad \text{for } x \in (t_{n+1}, t_n], \quad n = 1, 2, \dots \end{aligned} \quad (12)$$

It is straightforward to see that f_ϕ is the unique function $f : (-1, 1] \rightarrow \mathbf{C}$ for which $f|_{(0, 1]} = \phi$ and $U_\varepsilon^X f(x) = zf(x)$ for all $x \in (-1, 1]$. Let now ϕ be an element of $C^\infty[0, 1]$ such that the support of ϕ (i.e. the closure in $[0, 1]$ of the set $\{t : \phi(t) \neq 0\}$) is contained in the interval $(0, -t_2)$. It is clear that $f_\phi \in C^\infty(-1, 1]$. Using formula (11) and the asymptotics (10), for any $z \in \mathbf{C}$, $|z| < (1/2 + \varepsilon)^{n+1}$ one can verify that

$$\lim_{t \downarrow -1} f_\phi^{(j)}(t) = 0, \quad j = 0, 1, \dots, n. \quad (13)$$

Therefore putting $f_\phi(-1) = 0$, we see that $f_\phi \in X = C^n[-1, 1]$ and $U_\varepsilon^X f_\phi = z f_\phi$. Hence $\sigma(U_\varepsilon^X)$ contains $\overline{D}(n+1, \varepsilon)$ and any point of $D(n+1, \varepsilon)$ is an eigenvalue of U_ε^X of infinite multiplicity.

The second part of Proposition 2 follows from Ruelle's results on spectra of positive transfer operators, see [2], Theorem 2.5 and Exercise 2.9.

4 Spectral properties of the operator U_ε in spaces of analytic functions.

The spectral properties of the operator U_ε in spaces of analytic functions differ considerably depending on the choice of the space and on the values $\varepsilon = 1/2$ or $\varepsilon \neq 1/2$. Furthermore, not all of these properties are known yet.

Proposition 3. *Let $\varepsilon \in (0, 1/2)$, $q \in (1, 1/\varepsilon - 1)$ and X be the Hardy space $\mathcal{H}^2(0, q)$. Then the operator U_ε^X is nuclear. Moreover,*

- (i) *the eigenvalues $z_n = z_n(U_\varepsilon^X)$ and the eigenspaces $\mathcal{E}(z_n(U_\varepsilon^X), U_\varepsilon^X)$ do not depend on q ;*
- (ii) *the eigenvalues z_n satisfy the inequality*

$$|z_n| \leq 1.5c^n, \quad \text{where } c = c(\varepsilon) = \sqrt{1/2 + \sqrt{\varepsilon(1-\varepsilon)}} < 1. \quad (14)$$

Proof.

It is easy to show that for any $r > 1$

$$\alpha(r) = \sup_{|z|=r} |a_\varepsilon(z)| = \frac{1}{2}(1 - \varepsilon + r + \varepsilon r^2).$$

The function α is continuous and strictly increasing on the interval $(1, 1/\varepsilon - 1)$, and $\alpha(r) < r$ for any $r \in (1, 1/\varepsilon - 1)$. Put $q' = \alpha^{-1}(q) > q$. From the definition of the operator U_ε (3), it follows that U_ε is a linear continuous operator from $\mathcal{H}^2(0, q)$ to $\mathcal{H}^2(0, q')$ with norm less than or equal to $1 + \varepsilon\alpha(q)$. Thus U_ε^X is the composition of this operator and of the operator defining the embedding of $\mathcal{H}^2(0, q')$ into $\mathcal{H}^2(0, q)$, which is nuclear with s -numbers $(q/q')^n$. Therefore, the operator U_ε^X is nuclear with s -numbers $s_n(U_\varepsilon^X) \leq (1 + \varepsilon\alpha(q))(q/q')^n$.

From Remark 3 follows that the eigenvalues $z_n = z_n(U_\varepsilon^X)$ and the eigenspaces $\mathcal{E}(z_n(U_\varepsilon^X), U_\varepsilon^X)$ do not depend on q and coincide with the eigenvalues and eigenspaces of $U_\varepsilon^{C^\infty[-1, 1]}$.

From Weyl's inequality [24] we have

$$|z_n|^{n+1} \leq \prod_{k=0}^n |z_k| \leq \prod_{k=0}^n s_k(U_\varepsilon^X) \leq (1 + \varepsilon\alpha(q))^{n+1} (q/q')^{(n+1)n/2}$$

and therefore

$$z_n \leq (1 + \varepsilon\alpha(q))(q/q')^{n/2}. \quad (15)$$

The ratio q/q' is minimal for $q' = \sqrt{1/\varepsilon - 1}$ and is equal to $1/2 + \sqrt{\varepsilon(1 - \varepsilon)}$. For this value of q' we have $1 + \varepsilon\alpha(q) \leq 1 + \sqrt{\varepsilon - \varepsilon^2} \leq 1.5$. Therefore inequality (15) for $q' = \sqrt{1/\varepsilon - 1}$ implies (14). \square

The case $\varepsilon = 1/2$ is much more difficult and so far there exist very few results on the spectral properties of the Frobenius–Perron operators of the maps with parabolic neutral fixed points. We would like to point out the result of H. Rugh [25], who considered the Frobenius–Perron operators of piece-wise analytic maps, which are expanding everywhere except one parabolic fixed point. Namely, he constructed a specific map-dependent Banach space of analytic functions, where the spectrum of the F.P.O consists of the segment $[0,1]$ and some isolated normal eigenvalues. This space is in fact the image of $L_1[0, +\infty)$ with respect to some map-dependent integral transformation (similar to the Laplace transform). This idea applied to the cusp map allows to verify that the F.P.O. $U_{1/2}$ has similar spectral properties in certain weighted Hardy spaces in disks $D(\alpha, 1 + \alpha)$, $0 < \alpha < 1$.

The result of H. Rugh is very interesting since it provides the first example of a Banach space of smooth functions, where the spectrum of the Frobenius–Perron operator of the cusp map is non-trivial. Note that the functions of Rugh’s space are analytic in all points of the segment except the parabolic fixed point (-1 in our case). However we should notice that the spectrum of the F.P.O. of a map S in spaces of analytic functions with singularity at a fixed point of S may differ considerably from the spectrum in spaces of everywhere analytic functions. We illustrate this statement for the simplest expanding map F_0 , which is the tent map.

Proposition 4. *Let $p \in [1, +\infty]$, $0 < \alpha < 1$ and $X = \mathcal{H}^p(\alpha, 1 + \alpha)$. Then the spectrum $\sigma(U_0^X)$ depends on p . Namely, $\sigma(U_0^X)$ is the union of the disk $\overline{D}(0, 2^{2/p-1})$ and some set of (isolated) normal eigenvalues.*

Proof. Evidently, $U_0^X = A + B$, where $Af(x) = \frac{1}{2}f\left(\frac{1-x}{2}\right)$, $Bf(x) = \frac{1}{2}f\left(\frac{x-1}{2}\right)$. Since the image of the operator B is contained in the space $H^q(\alpha, \beta)$, where $\beta = \min\{1 + 5\alpha, 3 - \alpha\} > 1 + \alpha$, the operator B is nuclear and therefore compact.

Let us estimate now the norm of the operator A . Let $f \in X$. Then

$$\begin{aligned} \|Af\|^q &= \int_{D(\alpha, 1+\alpha)} \left(\frac{1}{2} \left| f\left(\frac{x+iy-1}{2}\right) \right| \right)^q dx dy = \int_{D(\alpha-1, (1+\alpha)/2)} \frac{4}{2^q} |f(x+iy)|^q dx dy \leq \\ &= \frac{1}{2^{q-2}} \int_{D(\alpha, 1+\alpha)} |f|^q dx dy = \frac{1}{2^{q-2}} \|f\|^q. \end{aligned}$$

Therefore $\|A\| \leq 2^{\frac{2}{q}-1}$. On the other hand one can verify that $Af_\lambda = 2^{-1-\lambda}f_\lambda$, where $f_\lambda(x) = (x+1)^\lambda$ and $f_\lambda \in X$ if and only if

$$\operatorname{Re} \lambda > -\frac{2}{q} \iff |2^{-1-\lambda}| \leq 2^{\frac{2}{q}-1}.$$

Hence, the open disk $D(0, 2^{\frac{2}{q}-1})$ is contained in the spectrum of A . Since $\|A\| \leq 2^{\frac{2}{q}-1}$, we find that $\sigma(A) = \overline{D}(0, 2^{\frac{2}{q}-1})$.

Since the operator B is compact and $U_0^X = A + B$, the theorem on holomorphic operator-functions ([21], Chapter I) implies that the spectrum of U_0^X is the union of $\overline{D}(0, 2^{\frac{2}{q}-1})$ and some (isolated) normal eigenvalues.

Proposition 5. *Let $0 < \nu < 0.3$ and X be the space of the functions $f : (-1, 1] \rightarrow \mathbf{C}$ such that the function $g_f(z) = f(-1 + 2^{-z})$, $g : [-1, +\infty) \rightarrow \mathbf{C}$ admits the analytic continuation to some element of the conventional Hardy Hilbert space Y in the half-plane $A_\nu = \{\operatorname{Re} z > -1 - \nu\}$ (We transfer the scalar product from this Hardy space to X by the bijective linear transform $f \mapsto g_f$). Then $\sigma(U_0^X) = [0, 1] \cup S$, where S consists of normal eigenvalues.*

Remark 4 The space X of Proposition 5 is a Hilbert space of functions analytic on the set $D(-1, c) \setminus (-1 - c, -1]$ for some $c = c(\nu) > 2$.

Proof of Proposition 5.

From the definition of the scalar product in X , the operator $T : X \rightarrow Y$, $Tf(x) = f(-1 + 2^{-x})$ is a unitary transformation. Therefore the operator

$$W = TU_0T^{-1} : Y \rightarrow Y$$

and U_0 are unitarily equivalent. From the definitions of T and U_0 it follows that $W = A + B$, where

$$Af(x) = \frac{1}{2}f(x+1), \quad Bf(x) = \frac{1}{2}f(-\log_2(2 + 2^{-y-1})).$$

It is straightforward to verify that the closure of the set $\{-\log_2(2 + 2^{-y-1}) : y \in A_\nu\}$ is a compact subset of A_ν . Hence, the operator B is nuclear and therefore compact. On the other hand the conventional Laplace transform and a linear change of variables provide a unitary equivalence between the operator A and the operator of multiplication with the function e^{-t} acting on a certain weighted Sobolev space of functions on $[0, +\infty)$. Therefore the spectrum of A is the segment $[0, 1]$.

Since the operator B is compact, the theorem on holomorphic operator-functions [21] implies that the spectrum of U_0^X , which is identical with the spectrum of W is the union of the segment $[0, 1]$ and some set of (isolated) normal eigenvalues. \square

It is worth noticing that the space constructed in Proposition 6 is obtained by the method similar to the construction of Rugh [25]. Thus, it is not a priori clear what is the origin of the “continuous spectrum” $[0, 1]$ obtained in [25]: the dynamical properties of the map or the choice of the space.

We conjecture that in the space of real-analytic on $[-1, 1]$ functions, the point spectrum of the Frobenius–Perron operator $U_{1/2}$ of the cusp map is $\{0, 1\}$, i.e., the eigenfunction equation $U_{1/2}f = zf$ has non-zero analytic solutions only for $z = 0$ and $z = 1$. To support this conjecture we show that $\{0, 1\}$ is the point spectrum of $U_{1/2}$ in the space of entire functions.

Proposition 6. *Let $\varepsilon = 1/2$, X be the space of entire functions. Then the spectrum of U_ε^X is the whole complex plane \mathbf{C} and the point spectrum of U_ε^X is the two-point set $\{0, 1\}$. The eigenvalue 0 has infinite multiplicity, and the eigenvalue 1 has multiplicity 1.*

Proof. The ergodicity of the map $F_{1/2}$ implies the multiplicity 1 for the eigenvalue $z = 1$. The null space of the operator $U_{1/2}$ is

$$\{f \in X : f(x) = (1+x)g(x) : g \text{ is an odd function.}\}$$

Therefore 0 is an eigenvalue of $U_{1/2}$ of infinite multiplicity. Let now $z \in \mathbf{C} \setminus \{0, 1\}$, $\psi \in X$ and $U_{1/2}\psi = z\psi$. The eigenvalue equation for $x = 1$ implies that $\psi(-1) = 0$. Therefore $\psi(x) = (1+x)g(x)$ for some $g \in X$. Let $\xi(x) = g(x) + g(-x)$. The eigenvalue equation $U_{1/2}\psi = z\psi$ in terms of the function ξ can be rewritten as

$$\xi\left(\left(\frac{x+1}{2}\right)^2\right) = \frac{32z\xi(x)}{x^3 + 5x^2 + 11x + 15} + \frac{x^3 - 5x^2 + 11x - 15}{x^3 + 5x^2 + 11x + 15}\xi\left(\left(\frac{x-1}{2}\right)^2\right). \quad (16)$$

Let $M(R) = \max_{|x|=R} |\xi(x)|$, $c \in (0, \sqrt{2})$. It is easy to see that if $x \in \mathbf{C}$, $\operatorname{Re}(x+1)^2 \geq 0$, $\operatorname{Re} x \geq 0$, and $R \leq |(x+1)^2/4| \leq R + c\sqrt{R}$ then, for sufficiently large $R > 0$, $|x| \leq R$ and $|(x-1)^2/4| \leq R$. Since ξ is even this fact together with formula (16) imply that

$$M(R + \sqrt{R} + 1/4) \leq M(R)(1 + 5/\sqrt{R} + O(1/R)) \quad \text{when } (R \rightarrow +\infty). \quad (17)$$

Applying (17) to $R_n = n^2/4$ and using the equality $R_{n+1} = R_n + \sqrt{R_n} + 1/4$, we obtain

$$M(n^2/4) \leq c_1 \prod_{k=1}^n (1 + 10/k + O(1/k^2))$$

for some positive constant c_1 . Therefore $M(n^2/4) = O(n^{10})$, and $M(R) = O(R^5)$. This estimation implies (see [26]) that ξ is a polynomial of degree at most 5. On the other hand, using induction with respect to the degree of polynomial, one can show that there are no polynomial solutions of the equation (16). \square

Remark 5. Similar technique allows verifying that for any $z \in \mathbf{C}$, $z \neq 0$ the function $f(x) = x$ does not belong to the space $U_{1/2}(X)$, where X is the set of all entire functions. Therefore, the spectrum of $U_{1/2}^X$ is the whole complex plane.

5 Numerical results for the spectra

In the previous section we presented the general description of the spectrum of the operator U_ε . However, the eigenvalues and the eigenfunctions of the cusp family are not known explicitly. So we should compute them numerically. In order to perform this calculation in the space of the analytic functions, we use Taylor's expansion:

$$f(x) = \sum_{k=0}^N c_k x^k. \quad (18)$$

The eigenvalue problem $U_\varepsilon f(x) = z f(x)$ can be reformulated in terms of the coefficients c_k :

$$U_\varepsilon f(x) = \sum_{k=0}^N c_k U_\varepsilon x^k = \sum_{k=0}^N c_k \sum_{p=0}^N a_{pk} x^p = z \sum_{k=0}^N c_k x^k. \quad (19)$$

As the operator U_ε is nuclear, we can project the last expression onto the subspace $\{x_k\}_{k=0}^N$. Now the eigenvalue problem can be written as $A\vec{c} = z\vec{c}$, where $\{A\}_{kp} = a_{kp}$, see (19).

The coefficients a_{pk} in (19) are equal to

$$a_{pk} = \begin{cases} (-1)^p f(\varepsilon, k, p), & k \text{ is even,} \\ (-1)^{p+1} 2\varepsilon f(\varepsilon, k+1, p), & k \text{ is odd,} \end{cases} \quad (20)$$

where the function $f(\varepsilon, k, p)$ is defined as

$$f(\varepsilon, k, p) = \frac{1}{2^k} \sum_{l=0}^p C_k^l C_k^{p-l} \varepsilon^l (1-\varepsilon)^{k-l}.$$

The most precise and convenient way for the calculation of the coefficients a_{pk} is the use of the recurrence relation:

$$a_{p,k+2} = \frac{1}{4} \left\{ (1-\varepsilon)^2 a_{p,k} + (2\varepsilon-2) a_{p-1,k} + (-2\varepsilon^2 + 2\varepsilon + 1) a_{p-2,k} - 2\varepsilon a_{p-3,k} + \varepsilon^2 a_{p-4,k} \right\}.$$

This representation is much more accurate than the numerical integration used in [14] hence it permits using longer expansion (18) without loss of accuracy.

It is worth noticing that the matrix A is non-symmetric. Up to 2×10^3 terms in the expansion (18) were used to get converged results. In order to check convergence, we use the trace formula for the operator U_ε . Namely, as for $\varepsilon \in [0, 1/2)$ the operator U_ε is nuclear, we can calculate its trace by using the Grothendieck-Fredholm formula (see for example [2, 27]):

$$\text{tr } U_\varepsilon = \sum_{n=0}^{\infty} z_n = \frac{1}{1/2 - \varepsilon} - \frac{2}{\sqrt{9 - 4\varepsilon(1-\varepsilon)}} \quad (21)$$

and compare this value with the numerical calculations.

In Fig. 1, ten maximal eigenvalues of the operator U_ε are presented. Because of very good convergence of our numerical method for small ε , the asymptotics of the z_n as $\varepsilon \rightarrow 0$ can be numerically calculated:

$$\frac{z_{n+1}}{z_n} = \frac{1}{4} + \left(2n - \frac{1}{2}\right) \varepsilon + O(\varepsilon^2). \quad (22)$$

Hence the cusp family has neither spectrum in the form r^n , where $n \in \mathbf{N}$, $r \in \mathbf{R}$, nor combination of few such spectra when $\varepsilon \neq 0$.

Using relation (22), we can find a general formula for the eigenvalues when ε is small:

$$z_{n+1} = \left(\frac{1}{4}\right)^n (1 + 2n(2n+1)\varepsilon + O(\varepsilon^2)), \quad n = 0, 1, 2, \dots \quad (23)$$

This result gives for the asymptotics of the trace

$$\text{tr } U_\varepsilon = \frac{4}{3} + \frac{104}{27} \varepsilon + O(\varepsilon^2) \quad \text{when } \varepsilon \rightarrow 0. \quad (24)$$

Formula (24) coincides with the asymptotics of Eq. (21). This coincidence supports strongly formulas (22,23) which are obtained only numerically.

When $\varepsilon \rightarrow 1/2$ and n is fixed, one can see that $z_n \rightarrow 1$. This result agrees with the divergence of the trace. We have also checked that the eigenvalues have the asymptotics

$$z_n = (1/2 + \varepsilon)^n \quad \text{when} \quad \varepsilon \rightarrow 1/2, \quad (25)$$

that agrees with the asymptotics found in [14].

Let us now discuss the eigenfunction behavior. In Figs. 2a, 2b we present the second and fourth eigenfunctions, respectively, for few values of ε . One can easily see a concentration effect in a vicinity of -1 as $\varepsilon \rightarrow 1/2$. The eigenfunctions tend to have the support only at the point $x = -1$. This behavior is in a good agreement with the existence of a “formal eigenfunction” $\delta(x + 1)$ for $\varepsilon = 0.5$. Such behavior of eigenfunctions supports numerically the conjecture about the non-existence of non-trivial (except of $\{0, 1\}$) spectrum for the cusp map in the space of the real analytic functions as the limit functions have a singularity at the point -1 .

6 Conclusions

The spectral properties of the cusp family (1) that “interpolates” between the tent map and the cusp map have been investigated in different function spaces. This study has permitted us to formulate a conjecture about the spectrum of the cusp map. While some results about this spectrum can be proved, the general description in different spaces of analytic functions is still unknown.

There are few questions which are particularly interesting in this context. First, the question about the asymptotics of the autocorrelation function for the cusp map. As the resonance eigenvalues tend to unity, one can expect non-exponential decrease of the autocorrelation function. The estimations in paper [15] show that the autocorrelation function $C(n)$ decreases as $1/n$ when $n \rightarrow \infty$. However, this conjecture is not yet analytically proven. Another question addresses the choice of the space of analytic functions where the spectrum of the F.P.O. is naturally defined by the dynamics of the map. Moreover, our calculations and calculations of [14] show that the spectrum of the cusp family is real. While there are some analytical results about a reality of the spectrum [28], they are not applicable to the cusp family. Hence the question about the reality of the spectrum also remains open.

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References

- [1] Bedford T, Keane M and Series C 1991 *Ergodic theory, symbolic dynamics, and hyperbolic spaces*, (Oxford University Press)
- [2] Baladi V 2000 *Positive Transfer Operators and Decay of Correlations*, (World Scientific)
- [3] Antoniou I and Tasaki S 1993 *Int. Journal of Quantum Chem.* **46** pp 425 – 474
- [4] Antoniou I, Dmitrieva L, Kuperin Yu and Melnikov Yu 1997 *Int. Journal Computers & Math. Applic.* **34** pp 399 – 425
- [5] Antoniou I and Shkarin S 1999 *Generalized functions, operator theory, and dynamical systems eds. Antoniou I and Lumer G* (Chapman & Hall/CRC) pp 171 – 201
- [6] Lasota A and Mackey M 1985 *Probabilistic Properties of Deterministic Systems* (Cambridge University Press, Cambridge, U.K.)
- [7] Cornfeld I P, Fomin S V and Sinai Ya G 1982 *Ergodic Theory* (Springer-Verlag, New York, Heidelberg, Berlin)
- [8] Antoniou I, Sadovnichii V A and Shkarin S A 1999 *Phys. Lett. A* **258** pp 237 – 243
- [9] Antoniou I and Qiao Bi 1996 *Phys. Lett. A* **215** pp 280 – 290
- [10] Hemmer P C 1984 *J. Phys. A* **17** pp L247 – L249
- [11] Ott E 1981 *Rev. Mod. Phys.* **53** pp 655 – 672
- [12] Tucker W 1999 *C. R. Acad. Sci. Paris* **328** ser 1 pp 1197–1202
- [13] Györgyi G and Szépfalusy P 1984 *Z. Phys. B* **55** pp 179
- [14] Kaufmann Z, Lustfeld H and Bene J 1996 *Phys. Rev. E* **53** pp 1416–1421
- [15] Lustfeld H and Szépfalusy P 1996 *Phys. Rev. E* **53** pp 5882–5889
- [16] Moon H T 1993 *Phys. Rev. E* **47** pp R772 – R775
- [17] Lasota A and Yorke J A 1973 *Trans. of AMS* **186** pp 481 – 489
- [18] Prellberg T and Slawny J 1992 *J. Stat. Phys.* **66** pp 503 – 514
- [19] Bruin 1995 *Comm. Math. Phys.* **168** pp 571 – 580
- [20] Matthew N 2001 *Discr. Cont. Dynamical Syst.* **7** pp 147 – 154
- [21] Gokhberg I C and Krein M G 1969 *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs, vol. 18, AMS

- [22] Robertson A and Robertson V 1964 *Topological Vector Spaces* (Cambridge University Press)
- [23] Edwards R E 1965 *Functional Analysis* (Holt, Rinehart and Winston)
- [24] Weyl H 1949 *Proc. Acad. Sci. USA* **35** pp 408–411
- [25] Rugh H H 1999 *Inventiones Mathematicae* **135** pp 1–24
- [26] Titchmarsh E C 1984 *The theory of functions* (Oxford University Press)
- [27] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner and G. Vattay, *Classical and Quantum Chaos*, www.nbi.dk/ChaosBook/, Niels Bohr Institute (Copenhagen 2001)
- [28] Rugh H H 1994 *Nonlinearity* **7** pp 1055–1066

Figure captions

Fig. 1. Ten maximal eigenvalues as the functions of ε .

Fig 2a, 2b. The second (2a) and fourth (2b) eigenfunctions for $\varepsilon = 0$ (the solid line), 0.25 (the long-dashed line), 0.4 (the dashed line), and 0.48 (the short-dashed line).





